

THE ERDŐS–PÓSA PROPERTY FOR ODD CYCLES IN HIGHLY  
CONNECTED GRAPHS

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*Dedicated to the memory of Paul Erdős*

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In [9] Thomassen proved that a  $2^{3^{9k}}$ -connected graph either contains  $k$  vertex disjoint odd cycles or an odd cycle cover containing at most  $2k - 2$  vertices, i.e. he showed that the Erdős–Pósa property holds for odd cycles in highly connected graphs. In this paper, we will show that the above statement is still valid for  $576k$ -connected graphs which is essentially best possible.

**1. Introduction**

A family  $\mathcal{F}$  of graphs is said to have the *Erdős–Pósa property*, if for every integer  $k$  there is an integer  $f(k, \mathcal{F})$  such that every graph  $G$  either contains  $k$  vertex disjoint subgraphs each isomorphic to a graph in  $\mathcal{F}$  or a set  $C$  of at most  $f(k, \mathcal{F})$  vertices such that  $G - C$  has no subgraph isomorphic to a graph in  $\mathcal{F}$ . The term *Erdős–Pósa property* arose because in [3] Erdős and Pósa proved that the family of cycles has this property. The family of odd cycles does not have the Erdős–Pósa property, as we now show. For a graph  $G$  an *odd cycle cover* is a set of vertices  $C \subseteq V(G)$  such that  $G - C$  is bipartite.

An elementary wall of height eight is depicted in Figure 1. An *elementary wall* of height  $h$  for  $h \geq 3$  is similar. It consists of  $h$  levels each containing  $h$  bricks, where a brick is a cycle of length six. A *wall* of height  $h$  is obtained

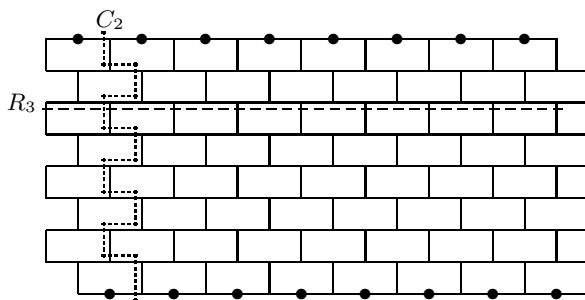
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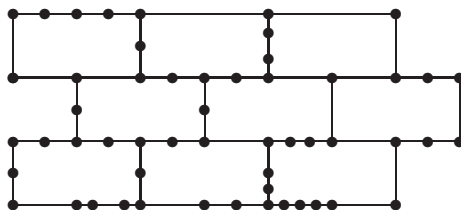
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**Fig. 1.** An elementary wall of height 8

from an elementary wall of height  $h$  by subdividing some of the edges, i.e. replacing the edges with internally vertex disjoint paths with the same end-points (see Fig. 2).



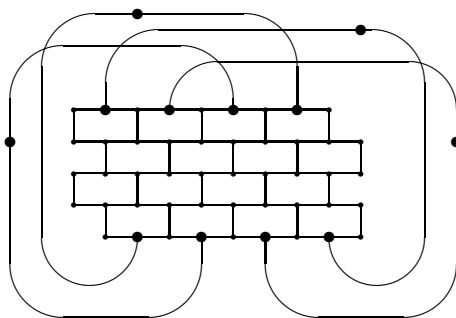
**Fig. 2.** A wall of height 3

An *Escher wall of height  $h$*  consists of a wall of height  $h$ :  $W$ , and  $h$  vertex disjoint paths  $P_1, \dots, P_h$  such that:

- (i) Each  $P_i$  has both endpoints on  $W$  but is otherwise disjoint from  $W$ .
- (ii) One endpoint of  $P_i$  is in the  $i$ th brick of the top row of bricks of  $W$ , the other is in the  $(h+1-i)$ th brick of the bottom row of  $W$ . Furthermore, both of these vertices are in only one brick of  $W$ .
- (iii)  $W$  is bipartite but for each  $i$ ,  $W \cup P_i$  contains an odd cycle.

See Figure 3 for an example.

We remark that, as pointed out by Lovász and Schrijver [5], an Escher wall  $W$  of height  $h$  contains neither 2 vertex disjoint odd cycles nor an odd cycle cover with fewer than  $h$  vertices (the first fact follows from the fact that for any Escher wall  $W, P_1, \dots, P_k$ , the planar embedding of  $W$  can be extended to an embedding of the Escher wall in the projective plane so that every odd cycle is non-null homotopic by routing the  $P_i$  through a cross-cap;



**Fig. 3.** An Escher wall of height 4

the fact that there is no odd cycle cover of size  $h-1$  follows from the fact that if  $X$  is a set of  $h-1$  vertices then  $X$  fails to intersect some path along the top of a level of bricks of  $W$  and then, as is easily verified, there is some  $i$  such that  $P_i$  is disjoint from  $X$  and both endpoints of  $P_i$  are connected to this row in  $W-X$ . Hence the two endpoints of this  $P_i$  are connected by a path  $Q$  in  $W-X$  and then  $P_i+Q$  is an odd cycle in  $G-X$ . This shows that the Erdős-Pósa property does not hold for odd cycles (in fact, it holds for the cycles of length  $p \bmod m$  if and only if  $p$  is congruent to 0  $\bmod m$  see [2] and [8]).

In [9] Thomassen proved, amongst other results, that for any integer  $k$  a  $2^{3^{9k}}$ -connected graph either contains  $k$  vertex disjoint odd cycles or an odd cycle cover of size at most  $2k-2$ . The size of the odd cycle cover in his result is best possible as can be seen by considering a large complete bipartite graph in which we add a complete graph on  $2k-1$  vertices in one of the partite sets. This construction shows that no condition on the connectivity can lead to smaller odd cycle covers in graphs without  $k$  vertex disjoint odd cycles.

In this paper we will improve on Thomassen's result showing that the above statement is still valid for  $576k$ -connected graphs. This is essentially best possible in the sense that we need to impose connectivity of  $\Omega(k)$  to obtain this result. Our approach is to split each vertex of a minimum odd cycle cover into two, thereby producing bipartite auxiliary graphs and then to analyze the set of paths in the auxiliary graphs corresponding to odd cycles in the original graph.

This approach was first adopted in [7]. In that paper Reed proved, amongst other results, that for all  $k$  there is an  $n_k$  such that a graph either contains a set of  $2k$  odd cycles such that every vertex is in at most two

of these odd cycles or an odd cycle cover of size at most  $n_k$ . Much of our introductory discussion was taken from that paper.

We now define precisely the auxiliary graphs we study. We define for every graph  $G$  and odd cycle cover  $C = \{c_1, \dots, c_l\}$  of  $G$ , a *first auxiliary (bipartite) graph*  $G' = G'(G, C)$ . To do so, we first choose an arbitrary partition of  $G - C$  into two stable sets  $A'$  and  $B'$  (note that we have only one choice, if  $G - C$  is connected). We set  $V(G') = (V(G) - C) \cup \{c^A, c^B \mid c \in C\}$  and  $E(G') = E(G - C) \cup \{c^A y \mid c \in C, y \in B', cy \in E(G)\} \cup \{c^B y \mid c \in C, y \in A', cy \in E(G)\} \cup \{c_i^A c_j^B \mid c_i, c_j \in C, c_i c_j \in E(G), i < j\}$ . We note that  $G'$  is bipartite with bipartition  $(A = A' \cup \{c^A \mid c \in C\}, B = B' \cup \{c^B \mid c \in C\})$ . For any vertex  $c$  in  $C$  we define  $image(c)$  as  $\{c^A, c^B\}$  and  $preimage(c^A) = preimage(c^B) = c$ . For any vertex  $x$  in  $G - C$ , we set  $image(x) = preimage(x) = x$ . For any set  $S$  of vertices of  $G$ , the *image* of  $S$  is the union of the images of its elements. Similarly, for any set  $S$  of vertices of  $G'$ , the *preimage* of  $S$  is the union of the preimages of its elements. Now, each edge  $e$  of  $G'$  corresponds to an edge of  $G$  whose endpoints are the preimages of the endpoints of  $e$ . We call this edge the *preimage* of  $e$ . The preimage function extends to subgraphs in a natural manner. For any vertex  $c \in image(C)$  we define the *partner*  $p(c)$  of  $c$  as the unique vertex different from  $c$  such that  $preimage(c) = preimage(p(c))$  and  $c$  and  $p(c)$  will be called *partners*. For any set  $S$  of vertices in  $image(C)$ , the set  $p(S)$  of partners of  $S$  is the union of the partners of its elements.

Now, for any subset  $X = \{x_1, \dots, x_k\}$  of  $C$ , if there are  $k$  vertex disjoint paths  $P_1, \dots, P_k$  in  $G' - image(C - X)$ , such that  $P_i$  links  $x_i^A$  with  $x_i^B$ , then the preimages of these paths form  $k$  vertex disjoint odd cycles of  $G$ . In the proof of our main result we will look for such sets of paths as well as similar more complicated structures using a second auxiliary bipartite graph  $G'' = G''(G')$ .

The graph  $G''$  arises from  $G'$  by deleting for all  $c \in C$  one of the two vertices in  $image(c)$ . If for any subset  $X = \{x_1, \dots, x_k\}$  of  $C$  we can find  $k$  vertex disjoint paths  $P'_1, \dots, P'_k$  in the graph  $G'[image(X) \cup V(G'')]$  such that  $P'_i$  links  $x_i^A$  and  $x_i^B$ , then these paths use at most one of the partners in  $image(c)$  for any  $c \in C - X$ . Hence the preimages of these paths will form  $k$  vertex disjoint odd cycles of  $G$ . We will find such paths using a highly connected subgraph  $F$  of  $G''$ . If  $G''$  has large average degree, then there are several results that imply the existence of such a highly connected subgraph. Since we are not interested in optimizing our constants, we use the following result due to Mader [6] which is one of the first and simplest of those results.

**Theorem 1 (Mader [6]).** *Every graph  $G$  of order  $n(G) \geq 2k - 1$  with more than  $(2k - 3)(n(G) - (k - 1))$  edges contains a  $k$ -connected subgraph.*

The next lemma shows that we can choose for all  $c \in C$  one of the partners in  $\text{image}(c)$  to be deleted from  $G'$  in such a way that  $G''$  has large average degree.

**Lemma 1.** *Let  $G$  be a  $576k$ -connected graph,  $C = \{c_1, c_2, \dots, c_l\}$  be an odd cycle cover of  $G$  and  $G'$  be the first auxiliary graph.*

*There is a partition  $\text{image}(C) = C_1 \cup C_2$  such that for  $1 \leq i \leq l$  the set  $C_1$  (and hence  $C_2$ ) contains exactly one of the vertices in  $\text{image}(c_i)$  and such that  $G'' = G' - C_2$  has average degree at least  $144k$ .*

**Proof.** For every  $c \in C$  we choose uniformly at random either  $c^A$  or  $c^B$  to belong to  $C_1$  and set  $C_2 = \text{image}(C) - C_1$  and  $G'' = G' - C_2$ . Let  $c \in C_1$  be any fixed vertex. The expected number of edges of  $G''$  that are incident with  $c$  and a vertex in  $V(G) - C$  is exactly half of the number of edges that are incident with  $\text{preimage}(c)$  and a vertex in  $V(G) - C$ . Furthermore, the expected number of edges of  $G''$  that are incident with  $c$  and another vertex in  $C_1$  is exactly a quarter of the number of edges that are incident with  $\text{preimage}(c)$  and another vertex in  $C$ . Let  $m_C$  and  $m_{C_1}$  denote the number of edges in  $G$  and  $G''$  incident with vertices in  $C$  and  $C_1$ , respectively. Now, by linearity of expectation,  $E[m_{C_1}] \geq \frac{1}{4}m_C$ . Hence there is a choice of  $C_1$  such that  $m_{C_1} \geq \frac{1}{4}m_C$ . This implies  $m(G'') \geq \frac{1}{4}m(G)$ . Since the minimum degree in  $G$  is at least  $576k$ , we have  $\frac{2m(G)}{n(G)} \geq 576k$ . As  $n(G) = n(G'')$ , this yields  $\frac{2m(G'')}{n(G'')} \geq 144k$ . ■

Applying [Theorem 1](#), we get the following.

**Corollary 1.** *Let  $G, C, G', C_1, C_2$  and  $G''$  be as in [Lemma 1](#), then  $G''$  contains a  $36k$ -connected subgraph  $F$ .*

During the proof we will have to link different pairs of vertices in  $F$  by vertex disjoint paths in  $F$ . The existence of such paths is guaranteed by the following result due to Bollobás and Thomason [1].

A graph  $G$  of order at least  $2k$  is  $k$ -linked, if for any two disjoint sets  $\{x_1, x_2, \dots, x_k\}$  and  $\{y_1, y_2, \dots, y_k\}$  of  $k$  vertices there are  $k$  vertex disjoint paths  $P_1, \dots, P_k$  such that for  $1 \leq i \leq k$  the path  $P_i$  has endpoints  $x_i$  and  $y_i$ . The following result ensures that a highly connected graph is also highly linked.

**Theorem 2** (Bollobás and Thomason [1]). *Every  $22k$ -connected graph is  $k$ -linked.*

## 2. Result

Having constructed the two auxiliary graphs we are now in a position to prove the theorem.

**Theorem 3.** *Let  $G$  be a  $576k$ -connected graph. Then  $G$  either contains  $k$  vertex disjoint odd cycles or an odd cycle cover of at most  $2k-2$  vertices.*

**Remark.** The constant 576 is certainly not best possible. We have made no attempt to minimize it.

**Proof.** Clearly  $k \geq 2$ . Let  $G$  be a  $576k$ -connected graph and let  $C$  be a minimum odd cycle cover of  $G$ . We assume that  $|C| \geq 2k-1$  and prove the existence of  $k$  vertex disjoint odd cycles in  $G$ . We consider two cases.

**Case 1.**  $|C| \leq 15k$

We prove that every  $500k$ -connected graph  $G$  with a minimum odd cycle  $C$  of size exactly  $2k-1$  contains  $k$  vertex disjoint odd cycles. The statement of the theorem then follows by deleting all but  $2k-1$  vertices from  $C$ .

Note that the graph  $G-C$  is  $450k$ -connected. Let  $A', B'$  be a bipartition of  $G-C$ . Since  $G$  is  $500k$ -connected, every vertex has degree at least  $500k$  and we can partition  $C$  in two sets  $C_A$  and  $C_B$  such that every vertex of  $C_A$  has at least  $200k$  neighbours in  $A'$  and every vertex in  $C_B$  has at least  $200k$  neighbours in  $B'$ . We may assume that  $C_A = \{c_1, \dots, c_{2l}\}$  and  $C_B = \{c_{2l+1}, \dots, c_{2k-1}\}$  for some  $0 \leq l \leq k-1$ .

If we can find a matching  $M_A$  in the graph  $G[C_A \cup B']$  that covers all of  $C_A$  and a matching  $M_B$  in  $G[C_B \cup A']$  that covers all of  $C_B$ , then we can find  $k$  vertex disjoint odd cycles in  $G$  as follows.

First, we choose different vertices  $a_1, \dots, a_{2l} \in A'$  and  $b_{2l+1}, \dots, b_{2k-1} \in B'$  such that for  $1 \leq i \leq 2l$  the vertex  $a_i$  is a neighbour of  $c_i$ , for  $2l+1 \leq i \leq 2k-1$  the vertex  $b_i$  is a neighbour of  $c_i$  and none of these vertices is incident with an edge in  $M_A \cup M_B$ . Such a choice is possible, since the vertices in  $C_A$  and  $C_B$  have at least  $200k$  neighbours in  $A'$  and  $B'$ , respectively.

For every vertex  $c_i$  in  $C_A$  that is matched to a vertex  $b$  in  $B$ , we can find an odd cycle through it by joining the vertices  $b$  and  $a_i$  by a path in  $G-C$ .

For every vertex  $c_i$  in  $C_A$  that is matched to a vertex  $c_j$  in  $C_A$ , we can find an odd cycle through these two vertices by joining their two neighbours  $a_i$  and  $a_j$  in  $A$  by a path in  $G-C$ . The cycles through vertices in  $C_B$  are constructed similarly.

We use at most  $2k-1$  paths in  $G-C$ . [Theorem 2](#) implies that we can choose these paths vertex disjoint. Since we need at most two vertices of  $C$  for one of these cycles, the graph  $G$  contains  $k$  vertex disjoint odd cycles.

Hence we can assume that there is no matching in  $G[C_A \cup B']$ , say, that covers all of  $C_A$ . Using a small extension of Tutte's matching theorem (see for instance [4]), this implies the existence of a set  $Z$  of vertices such that the graph  $G[C_A \cup B'] - Z$  has at least  $|Z| + 1$  odd components that are contained in  $C_A$ . Let the set  $Z'$  contain exactly one vertex from each of these components. Then the graph  $H = G - ((C - Z') \cup Z)$  is bipartite with bipartition  $((A \cap V(H)), ((B \cup Z') \cap V(H)))$ . Hence the set  $(C - Z') \cup Z$  is an odd cycle cover of size  $|C| - |Z'| + |Z| < |C|$  which contradicts the choice of  $C$ . This completes the proof of this case.

**Case 2.**  $|C| > 15k$ .

We consider the first auxiliary graphs  $G'$  choose a partition  $\text{image}(C) = C_1 \cup C_2$  and the second auxiliary graph  $G''$  containing a  $36k$ -connected subgraph  $F$  according to Lemma 1 and Corollary 1. For a set  $X \subseteq C$  let  $G'_X = G'[V(G'') \cup \text{image}(X)]$ .

We choose a maximum size set  $X \subseteq C$  such that there is no set of fewer than  $2|X|$  vertices separating  $\text{image}(X)$  and  $F$  in  $G'_X$  (note that in a graph  $H$  a set  $S \subseteq V(H)$  is said to *separate* two sets  $A, B \subseteq V(H)$ , if in  $H - S$  there is no path joining a vertex in  $A$  and a vertex in  $B$ ).

By Menger's Theorem, there are  $2|X|$  vertex disjoint paths joining  $\text{image}(X)$  to  $F$  in  $G'_X$  which are internally disjoint from  $F$ . If  $r := |X| \geq k$ , then we choose  $k$  vertices  $x_1, \dots, x_k \in X$ . For  $1 \leq i \leq k$  let  $f_i^A$  and  $f_i^B$  be the endpoints of the paths joining  $x_i^A$  and  $x_i^B$  to  $F$ , respectively. Let  $P_i$  be a path from  $f_i^A$  to  $f_i^B$  in  $F$ . Since  $F$  is  $k$ -linked, we can assume that the paths  $P_1, \dots, P_k$  are vertex disjoint. For  $1 \leq i \leq k$  we can now join the path from  $x_i^A$  to  $F$ ,  $P_i$  and the path from  $x_i^B$  to  $F$  and get  $k$  vertex disjoint paths that are internally disjoint from  $C_2$  and which correspond to  $k$  vertex disjoint odd cycles in  $G$ . Hence  $r < k$ .

We choose a set  $Z_0$  of size  $2r$  that separates  $\text{image}(X)$  and  $F$  in  $G'_X$  such that the component  $U_0$  of  $G'_X - Z_0$  that contains  $F - Z_0$  is as small as possible (note that  $F - Z_0$  is still at least  $34k$ -connected.)

**(2.1)** There is a set  $Z^*$  of size at most  $7k$  such that  $Z_0 \subseteq Z^*$  and the component  $U^*$  of  $G'_X - Z^*$  that contains  $F - Z^*$  contains no vertex in  $C_1$  whose partner is adjacent to a vertex in  $U^*$ .

**Proof.** Let  $x_1, x_2, \dots, x_j \in C_1$  be the vertices in  $U_0$  whose partners are adjacent to a vertex in  $U_0$ . We assume  $j > 0$ , since otherwise we can choose  $Z_0 = Z^*$ .

By the choice of  $X$ , for  $1 \leq i \leq j$  there is a set of at most  $2r+1$  vertices that separates  $\text{image}(X) \cup \{x_i, p(x_i)\}$  and  $F$  in  $G'_{X \cup \{\text{preimage}(x_i)\}}$ . We choose one

such set  $Z_i$  such that the component  $U_i$  of  $G'_{X \cup \{preimage(x_i)\}} - Z_i$  that contains  $F - Z_i$  is as small as possible. Note that this choice implies  $|Z_i| = 2r + 1$ .

**(2.1.1)**  $p(x_i) \notin Z_i$  for  $1 \leq i \leq j$ .

**Proof.** We assume that  $p(x_i) \in Z_i$  for some  $1 \leq i \leq j$  and derive a contradiction to the choice of  $Z_0$ . Let  $Z'_i = Z_i - \{p(x_i)\}$  and let  $U'_i$  be the component of  $G'_X - (Z_0 \cup Z'_i)$  that contains  $F - (Z_0 \cup Z'_i)$ . Let  $A_1$  be the set of vertices in  $Z_0 \cup Z'_i$  that are adjacent to a vertex in  $U'_i$  and let  $A_2$  be the set of vertices in  $Z_0 \cup Z'_i$  that are either in  $image(X)$  or adjacent to a vertex in a component of  $G'_X - (Z_0 \cup Z'_i)$  containing a vertex of  $image(X)$ . Since  $U'_i \subseteq U_0$ ,  $U'_i \subseteq U_i$  and  $Z_0$  and  $Z'_i$  separate  $image(X)$  and  $F$  in  $G'_X$ , we obtain that  $A_1 \cap A_2 \subseteq Z_0 \cap Z'_i$ . This implies  $|A_1| + |A_2| \leq |Z_0| + |Z'_i| = 2r + (2r + 1 - 1) = 4r$ . As  $A_2$  separates  $image(X)$  and  $F$  in  $G'_X$ , we have  $|A_2| \geq 2r$ , by the choice of  $X$ , and therefore  $|A_1| \leq 2r$ . Now  $A_1$  separates  $image(X)$  and  $F$  in  $G'_X$  and since  $U'_i \subseteq U_0$  and  $x_i \in U_0 - U'_i$  we obtain a contradiction to the choice of  $Z_0$ . ■

By (2.1.1), we see that  $Z_i$  separates

$$image(X) \cup \{x_i\} \cup N(p(x_i), G'_{X \cup \{preimage(x_i)\}}) \quad \text{and} \quad F \text{ in } G'_X.$$

Further, by definition, the component  $U_i$  of  $G'_X - Z_i$  is as small as possible given this condition.

**(2.1.2)** For  $1 \leq i \leq j$  we have  $Z_i \subseteq Z_0 \cup U_0$  and  $Z_i$  separates  $Z_0$  and  $F$  in  $G'_X$ .

**Proof.** As in the proof of (2.1.1), let  $U'_i$  be the component of  $G'_X - (Z_0 \cup Z_i)$  that contains  $F - (Z_0 \cup Z_i)$ . We choose the sets  $A_1$  and  $A_2$  as above and get  $|A_1| + |A_2| \leq |Z_0| + |Z_i| = 4r + 1$  and  $|A_2| \geq 2r$ . Hence  $|A_1| \leq 2r + 1$ . Since  $A_1$  separates  $Z_i$  and  $F$  in  $G'_X$ , we obtain, by the choice of  $Z_i$ , that  $A_1 = Z_i$  and (2.1.2) is proved. ■

**(2.1.3)** For any  $1 \leq s < t \leq j$  either  $Z_s = Z_t$  or  $Z_0 \subseteq Z_s \cup Z_t$ .

**Proof.** We assume that  $Z_s \neq Z_t$ . Note that  $U_s$  contains a vertex of  $Z_t$  and  $U_t$  contains a vertex of  $Z_s$ . Otherwise, we assume that  $U_s \cap Z_t = \emptyset$  which implies that  $Z_s$  separates  $Z_t$  from  $F$  in  $G'_X$  and therefore  $U_s \subseteq U_t$ . As  $Z_s \neq Z_t$ , we have  $U_s \neq U_t$  and we get a contradiction to the choice of  $Z_t$ . Let  $U_{s,t}$  be the component of  $G'_X - (Z_s \cup Z_t)$  that contains  $F - (Z_s \cup Z_t)$ . Let  $A_1$  be the set of vertices in  $Z_s \cup Z_t$  that are adjacent to a vertex in  $U_{s,t}$  and let  $A_2$  be the set of vertices that are in  $Z_0$  or are adjacent to a vertex in a component of  $G'_X - (Z_s \cup Z_t)$  containing a vertex of  $Z_0$ . Since  $Z_s$  and  $Z_t$  separate  $Z_0$  from  $F$  in  $G'_X$ , we have that  $A_1 \cap A_2 \subseteq Z_s \cap Z_t$  and therefore  $|A_1| + |A_2| \leq |Z_s| + |Z_t| = 4r + 2$ . As  $U_{s,t}$  is strictly contained in  $U_s$  and  $U_t$  and

$A_1$  separates  $Z_s$  and  $Z_t$  from  $F$  in  $G'_X$ , we see that  $|A_1| \geq 2r+2$  or we get a contradiction to the choice of  $Z_s$  or  $Z_t$ . Hence  $|A_2| \leq 2r$  and, as  $A_2$  separates  $Z_0$  and  $F$  in  $G'_X$ , we get  $A_2 = Z_0$ , by the choice of  $Z_0$ . This proves (2.1.3). ■

We may assume that  $Z_0 \subseteq Z_i$  for  $1 \leq i \leq j'$  and that  $Z_0 - Z_i \neq \emptyset$  for  $j' + 1 \leq i \leq j$ . For every  $1 \leq i \leq j'$  there is a vertex  $y_i \in U_0$  such that  $Z_i = Z_0 \cup \{y_i\}$ . Furthermore, we may assume that all sets  $Z_{j'+1}, \dots, Z_{j''}$  are different and that for every  $j'' + 1 \leq i \leq j$  there is some  $j' + 1 \leq i' \leq j''$  such that  $Z_i = Z_{i'}$ .

**(2.1.4)**  $G$  contains  $k$  vertex disjoint odd cycles or  $|\{y_1, y_2, \dots, y_{j'}\}| \leq k - 1$ .

**Proof.** We consider the block-structure of  $U_0$ . There is one block  $B_0$  of  $U_0$  that contains  $F - Z_0$ . By the choice of  $Z_i$ , we see that  $y_i$  is a cutvertex in  $B_0$  such that  $\{y_i\}$  separates  $\{x_i\} \cup N(p(x_i), G'_{X \cup \{preimage(x_i)\}})$  and  $F$  in  $U_0$ . This implies that the vertices in  $\{x_i\} \cup N(p(x_i), G'_{X \cup \{preimage(x_i)\}})$  are contained in one component of  $U_0 - \{y_i\}$ . We assume now that  $|\{y_1, y_2, \dots, y_{j'}\}| \geq k$ . Then there is a set  $Y$  of  $k$  cutvertices in  $B_0$  such that there are  $k$  components in  $U_0 - Y$  each of which contains a path from some  $x_i$  to a neighbour of its partner. Hence there are  $k$  vertex disjoint paths in  $G'$  from some  $x_i$ 's to their partners and all these paths are internally disjoint from  $C_2$ . Thus there is a set of  $k$  vertex disjoint odd cycles in  $G$ . This completes the proof of (2.1.4). ■

We may assume now that  $|\{y_1, y_2, \dots, y_{j'}\}| \leq k - 1$ . For all  $1 \leq i \leq j$  we have that  $|Z_i - Z_0| = |Z_0 - Z_i| + 1$  and since, by (2.1.3), every vertex of  $Z_0$  is missing from at most one  $Z_i$  for  $j' + 1 \leq i \leq j''$ , we have  $j'' - j' \leq 2r < 2k$ . This implies

$$\begin{aligned} |\cup_{i=0}^j Z_i| &\leq |Z_0| + |\{y_1, y_2, \dots, y_{j'}\}| + \sum_{i=j'+1}^{j''} |Z_i - Z_0| \\ &< 2k + k + 2k + \sum_{i=j'+1}^{j''} |Z_0 - Z_i| \leq 7k. \end{aligned}$$

Hence the set  $Z^* := \cup_{i=0}^j Z_i$  satisfies  $|Z^*| \leq 7k$ ,  $Z_0 \subseteq Z^*$  and the component  $U^*$  of  $G'_X - Z^*$  that contains  $F - Z^*$  contains no vertex in  $C_1$  whose partner is adjacent to a vertex of  $U^*$ . This proves (2.1). ■

Let  $X^* = X \cup preimage(\{x_1, \dots, x_j\})$ . The set  $Z^*$  separates  $image(X^*)$  and  $F$  in  $G'_{X^*}$ . Otherwise there is a path  $P$  in  $G'_{X^*} - Z^*$  from  $u \in image(X^*)$  to  $F$  that is internally disjoint from  $image(X^*)$ . Since  $u \in \{x_i, p(x_i)\}$  for some  $1 \leq i \leq j$  and  $Z_i \subseteq Z^*$ , the path  $P$  is contained in  $G'_{X \cup preimage\{x_i\}} - Z_i$ . This is a contradiction to the choice of  $Z_i$ .

Let  $C_1^* = U^* \cap C_1$ . By the construction of  $Z^*$ , the vertices in  $p(C_1^*) \subseteq C_2$  are not adjacent to a vertex in  $U^*$ .

**(2.2)**  $|C_1^*| \leq 7k$ .

**Proof.** We can partition the components of  $G' - (\text{image}(C) \cup Z^* - (C_1^* \cup p(C_1^*)))$  in two sets in such a way that for every vertex in  $C_1^*$  its partner lies in a component on the other side of this partition. Hence we can find a bipartition of  $G' - (\text{image}(C) \cup Z^* - (C_1^* \cup p(C_1^*)))$  such that  $G - (C \cup \text{preimage}(Z^*) - \text{preimage}(C_1^*))$  arises by identifying vertices on the same side of this bipartition and is therefore bipartite. Hence  $C \cup \text{preimage}(Z^*) - \text{preimage}(C_1^*)$  is an odd cycle cover of  $G$  of size at most  $|C| + |Z^*| - |C_1^*|$ . By the choice of  $C$ , we have  $|C| \leq |C| + |Z^*| - |C_1^*|$  which yields (2.2). ■

Let  $Z^{**} = Z^* \cup C_1^*$  and let  $U^{**}$  be the component of  $G'_{X^*}$  that contains  $F - Z^{**}$ .

**(2.3)** There are  $16k$  vertex disjoint paths from  $C_2 - \text{image}(X^*)$  to  $F$  in  $G' - Z^{**}$ .

**Proof.** If there are no such paths, then there is a set  $W$  of size less than  $16k$  that separates  $C_2 - \text{image}(X^*)$  and  $F$  in  $G' - Z^{**}$ . Since  $|W \cup Z^{**}| < 16k + 14k = 30k < |\text{image}(C)|$ , the set  $W \cup Z^{**}$  is a cutset of  $G'$  such that the component of  $G' - (W \cup Z^{**})$  that contains  $F - (W \cup Z^{**}) \neq \emptyset$  contains no vertex of  $\text{image}(C) - (W \cup Z^{**}) \neq \emptyset$ . Hence  $\text{preimage}(W \cup Z^{**})$  is a cutset of  $G$  of size at most  $30k$  which is a contradiction to the connectivity of  $G$ . ■

Hence there is a set of  $16k$  vertex disjoint paths from  $C_2 - \text{image}(X^*)$  to  $F$  in  $G' - Z^{**}$ . If we choose these paths minimal, then there is a set  $Y$  with  $Y \subseteq C_2 - \text{image}(X^*)$  of  $16k$  vertices which are endpoints of these paths and the paths are internally contained in  $U^{**}$  and internally disjoint from  $F$ . Let  $Y = \{y_1, \dots, y_{16k}\}$  and for  $1 \leq i \leq 16k$  let  $f_i$  be the endpoint of the path that joins  $y_i$  to  $F$ .

**(2.4)** There are  $|Y|$  vertex disjoint paths from  $Y$  to  $p(Y)$  in  $G'_{\text{preimage}(Y)}$ .

**Proof.** If there are no such paths, then there is a set  $W'$  of size less than  $|Y|$  that separates  $Y$  and  $p(Y)$  in  $G'_{\text{preimage}(Y)}$ . As above, we can now partition the components of  $G' - ((\text{image}(C) - (Y \cup p(Y))) \cup W')$  in two sets in such a way that for every vertex in  $Y$  its partner lies in a component on the other side of this partition. Hence  $G - ((C - \text{preimage}(Y)) \cup \text{preimage}(W'))$  is bipartite and  $(C - \text{preimage}(Y)) \cup \text{preimage}(W')$  is an odd cycle cover of  $G$  of size less than  $|C| - |Y| + |Y| = |C|$  which is a contradiction. ■

By the last claim, there is a set of  $2k$  vertex disjoint paths from  $Y$  to  $p(Y)$  in  $G'_{\text{preimage}(Y)} - Z^{**}$ . If we choose these paths  $P_1, \dots, P_{2k}$  minimal, then

they are internally disjoint from  $Y \cup p(Y) \cup U^{**}$ . For  $1 \leq i \leq 2k$  let the path  $P_i$  join  $p(y_{s_i}) \in p(Y)$  to  $y_{t_i} \in Y$ . We can now define  $k$  vertex disjoint paths  $P'_1, \dots, P'_k$  in  $G'$  that correspond to vertex disjoint odd cycles in  $G$ .

Let  $P'_1$  consist of the path  $P_1$  from  $p(y_{s_1})$  to  $y_{t_1}$ , the path from  $y_{t_1}$  to  $f_{t_1}$ , a path from  $f_{t_1}$  to  $f_{s_1}$  contained in  $F - Z^{**}$  and the path from  $f_{s_1}$  to  $y_{s_1}$ . Proceeding in this manner, we see that for one of the paths  $P'_i$  we have to use at most two of the  $2k$  paths  $P_j$ . Since  $F - Z^{**}$  is  $(36 - 14)k = 22k$ -connected and, by Theorem 2,  $k$ -linked, we can choose the  $k$  paths that are contained in  $F - Z^{**}$  to be vertex disjoint. We have therefore proved the existence of  $k$  vertex disjoint odd cycles in  $G$  and the proof of the theorem is completed. ■

A result similar to Theorem 3 cannot be obtained, if the condition on the connectivity is replaced by a condition on the minimum degree. To see this, identify each vertex  $v$  of an Escher wall with a vertex in a large complete bipartite graph  $G_v$ .

Furthermore, to obtain Theorem 3, we have to impose a connectivity of  $\Omega(k)$  on the graph which can be seen by the following example. Take a graph  $G$  with no two vertex disjoint odd cycles and no odd cycle cover of size  $2k - 1$ . Add  $\frac{k}{2}$  vertices that are joined to each vertex of  $G$ . The constructed graph is at least  $\frac{k}{2}$ -connected, has at most  $\frac{k}{2} + 1$  vertex disjoint odd cycles but has no odd cycle cover of size  $2k - 1$ . In view of this construction, it is conceivable that the statement of Theorem 3 holds for  $k$ -connected graphs.

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